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On dual quermassintegrals of mixed intersection bodies

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Abstract

In this paper, it is shown that a family of inequalities involving mixed intersection bodies holds. The Busemann intersection inequality is the first of this family. All of the members of this family are strengthened versions of classical inequalities between pairs of dual quermassintegrals of a star body.

Keywords: Busemann intersection inequality; Mixed intersection bodies; Dual quermassintegrals; Mixed projection bodies

1. Introduction and main results

Intersection bodies were first explicitly defined and named by Lutwak [7]. Following Lutwak, the intersection body of order i of a star body was introduced by Zhang [15]. The duality between intersection bodies (mixed intersection body) and projection bodies (mixed projection body) was first made clear in [7]. In recent years, several authors including Gardner [1–3], Koldobsky [4,5], Lutwak [7–9], Zhang [15] and Zhao [16] have given considerable attention to this duality.

One of the major outstanding questions in the area is Petty's conjectured inequality [13] between the volume of a convex body, i.e. a compact, convex set with non-empty interior in $\mathbb{R}^n$ and that of its projection body: For a convex body $K \subseteq \mathbb{R}^n$,

$$\omega_n^{n-2} W_0(\Pi_0 K) \geqslant \omega_{n-1}^n W_0(K)^{n-1}. \tag{P}$$

The classical isoperimetric inequality between the volume and surface area of a convex body states that, for a convex body $K \subseteq \mathbb{R}^n$,

$$W_1(K)^n \geqslant \omega_n W_0(K)^{n-1}.$$

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Petty's conjectured inequality is a strengthened version of the classical isoperimetric inequality. Specifically, the conjectured inequality (P) is that, for a convex body $K \subseteq \mathbb{R}^n$,

$$W_1(K)^n \geqslant \left(\frac{\omega_n^{n-1}}{\omega_{n-1}^n}\right) W_0(\Pi_0 K) \geqslant \omega_n W_0(K)^{n-1}. \tag{P_0}$$

E. Lutwak [10] conjectured the following strengthened version of inequality (P) between pairs of quermassintegrals of a convex body: For a convex body $K \subseteq \mathbb{R}^n$ and $0 \leqslant i < n-1$,

$$W_{i+1}(K)^{n-i} \geqslant \left(\frac{\omega_n^{n-1-i}}{\omega_{n-1}^{n-i}}\right) W_i(\Pi_i K) \geqslant \omega_n W_i(K)^{n-i-1}. \tag{P_i}$$

Petty's conjectured inequality (P) is the right inequality for $i = 0$ in (P_i). Lutwak [10] proved that if (P_0) is correct then so is (P_i), for $0 \leqslant i < n-1$ and that the conjectured inequality (P_i) is correct for $i = n-2$.

According to the duality, we consider the dual forms of inequality (P), (P_0) and (P_i). The dual version of Petty's conjectured inequalities (P) is that: For a star body $K \subseteq \mathbb{R}^n$,

$$\omega_n^{n-2} \tilde{W}_0(I_0 K) \leqslant \omega_{n-1}^n \tilde{W}_0(K)^{n-1}. \tag{B}$$

The purpose of this paper is to establish these dual inequalities.

Theorem 1. *If K is a star body in $\mathbb{R}^n$, then*

$$\tilde{W}_1(K)^n \leqslant \left(\frac{\omega_n^{n-1}}{\omega_{n-1}^n}\right) \tilde{W}_0(I_0 K) \leqslant \omega_n \tilde{W}_0(K)^{n-1}, \tag{B_0}$$

and there is equality in the left inequality if and only if K is a ball, and equality in the right inequality if and only if K is a centered ellipsoid.

Generalized versions of inequality (B_0) are obtained:

Theorem 2. *If K is a star body in $\mathbb{R}^n$ and $0 \leqslant i < n-1$, then*

$$\tilde{W}_{i+1}(K)^{n-i} \leqslant \left(\frac{\omega_n^{n-1-i}}{\omega_{n-1}^{n-i}}\right) \tilde{W}_i(I_i K) \leqslant \omega_n \tilde{W}_i(K)^{n-i-1}. \tag{B_i}$$

In Section 2, we give the necessary notation, definitions and background material. For reference see Gardner [3] and Schneider [14]. We shall prove these theorems in Section 3.

The ideas and techniques of Lutwak [10] play a critical role throughout this paper.

2. Notation and preliminaries

Let $\mathcal{K}^n$ denote the set of convex bodies in Euclidean space $\mathbb{R}^n$, for the set of convex bodies containing the origin in their interiors in $\mathbb{R}^n$, write $\mathcal{K}_o^n$. Let $V(K)$ denote the n-dimensional volume of body K. Let S^{n-1} denote the unit sphere in $\mathbb{R}^n$, ω_n denote the n-dimensional volume of the unit ball, D, in $\mathbb{R}^n$. For $u \in S^{n-1}$, $K \cap u^\perp$ denotes the intersection of K with the subspace $u^\perp$ that passes through the origin and is orthogonal to u, $K|u^\perp$ denotes the image of the orthogonal projection of K onto the hyperplane $u^\perp$.

If $K \in \mathcal{K}^n$, then its support function, $h_K = h(K, \cdot): \mathbb{R}^n \to \mathbb{R}$, is defined by

$$h(K, x) = \max\{x \cdot y\colon\ y \in K\}, \quad x \in \mathbb{R}^n, \tag{2.1}$$

where $x \cdot y$ denotes the standard inner product of x and y. The Hausdorff distance, $\delta(K, L)$, between $K, L \in \mathcal{K}^n$, can be defined by $\delta(K, L) = |h_K - h_L|_\infty$, where $|\cdot|_\infty$ is the sup-norm on the space of continuous functions, $C(S^{n-1})$.

Associated with a compact subset K of $\mathbb{R}^n$, which is star-shaped (about the origin), is its radial function, $\rho_K = \rho(K, \cdot): \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$, defined by

$$\rho(K, x) = \max\{\lambda \geqslant 0\colon\ \lambda x \in K\}, \quad x \in \mathbb{R}^n \setminus \{0\}. \tag{2.2}$$

If ρ_K is positive and continuous, K will be called a star body (about the origin). Let $\mathcal{S}^n$ denote the set of star bodies (about the origin) in $\mathbb{R}^n$.

For $K_1, \dots, K_n \in \mathcal{K}^n$, their mixed volume will be denoted by $V(K_1, \dots, K_n)$. If $K_1 = \dots = K_{n-i} = K$ and $K_{n-i+1} = \dots = K_n = L$, then the mixed volume $V(K_1, \dots, K_n)$ is written as $V_i(K, L)$. If $L = D$, then $V_i(K, D)$ is called the ith quermassintegral of K and is denoted by $W_i(K)$, $0 \leqslant i \leqslant n$. In particular, $W_0(K) = V(K)$, and $W_n(K) = \omega_n$.

Let $K_j \in \mathcal{S}^n$ $(1 \leqslant j \leqslant n)$. The dual mixed volume $\tilde{V}(K_1, \dots, K_n)$ is defined by (see [11])

$$\tilde{V}(K_1, \dots, K_n) = \frac{1}{n} \int\limits_{S^{n-1}} \rho(K_1, u) \cdots \rho(K_n, u)\, dS(u),$$

where dS is the $(n-1)$-dimensional volume element on S^{n-1}. If s, t are nonnegative integers whose sum does not exceed n, K, L are star bodies and $\mathcal{C}$ is the $(n-s-t)$-tuple of star bodies $(C_1, \dots, C_{n-s-t})$, then $\tilde{V}(K, s; L, t; \mathcal{C})$ will be used to denote the dual mixed volume $\tilde{V}(K, \dots, K, L, \dots, L, C_1, \dots, C_{n-s-t})$ in which K appears s times and L appears t times.

The dual Aleksandrov–Fenchel inequality (see [11]) states that

$$\tilde{V}(K, s; L, t; C)^{s+t} \leqslant \tilde{V}(K, s+t; C)^s \tilde{V}(L, s+t; C)^t, \tag{2.3}$$

with equality if and only if $K, L, C_1, \dots, C_{n-s-t}$ are dilations of each other. The dual Minkowski inequality states that (see [11]): If $K_j \in \mathcal{S}^n$ $(1 \leqslant j \leqslant n)$, then

$$\tilde{V}(K_1, \dots, K_n)^n \leqslant V(K_1)V(K_2) \cdots V(K_n) \tag{2.4}$$

with equality if and only if the star bodies are dilations of each other.

For $K \in \mathcal{S}^n$ and $i \in \mathbb{R}$, the ith dual quermassintegral, $\tilde{W}_i(K)$ of K is defined by (see Lutwak [11])

$$\tilde{W}_i(K) = \frac{1}{n} \int\limits_{S^{n-1}} \rho(K, u)^{n-i}\, dS(u). \tag{2.5}$$

In particular, $\tilde{W}_0(K) = V(K)$, and $\tilde{W}_n(K) = \omega_n$.

For $K \in \mathcal{K}^n$, the ith projection body of K, $\Pi_i K$, is the centrally symmetric convex body whose support function, for $u \in S^{n-1}$, is given by

$$h(\Pi_i K, u) = w_i\bigl(K \mid u^\perp\bigr), \tag{2.6}$$

where $w_i(K|u^\perp)$ is the ith projection measure of $K|u^\perp$ in $u^\perp$, and is called the $(n-i-1)$-girth of K in the direction u (see [12]).

If $K_1, \dots, K_{n-1}$ are star bodies in $\mathbb{R}^n$ and $u \in S^{n-1}$, then the $(n-1)$-dimensional dual mixed volume of $K_1 \cap u^\perp, \dots, K_{n-1} \cap u^\perp$ in $u^\perp$ is written $\tilde{v}(K_1 \cap u^\perp, \dots, K_{n-1} \cap u^\perp)$. If $K_1 = \dots = K_{n-i-1} = K$ and $K_{n-i} = \dots = K_{n-1} = D$, then $\tilde{v}(K_1 \cap u^\perp, \dots, K_{n-1} \cap u^\perp)$ is just the ith dual quermassintegral of $K \cap u^\perp$ in $u^\perp$, it will be denoted by $\tilde{w}_i(K \cap u^\perp)$ and is called the $(n-i-1)$-section of K in the direction u. The $(n-1)$-dimensional volume of $K \cap u^\perp$ will be written $v(K \cap u^\perp)$, rather than $\tilde{w}_0(K \cap u^\perp)$.

The intersection body of a star body K, IK, is the centrally symmetric body whose radial function on S^{n-1} is given by (see [7])

$$\rho(IK, u) = v\bigl(K \cap u^\perp\bigr). \tag{2.7}$$

If $K_1, \dots, K_{n-1} \in \mathcal{S}^n$, then the mixed intersection body $I(K_1, \dots, K_{n-1})$ is the centrally symmetric body (see [6]) whose radial function, for $u \in S^{n-1}$, is given by

$$\rho\bigl(I(K_1, \dots, K_{n-1}), u\bigr) = \tilde{v}\bigl(K_1 \cap u^\perp, \dots, K_{n-1} \cap u^\perp\bigr). \tag{2.8}$$

The mixed intersection bodies $I(K, \dots, K, L, \dots, L)$, with i copies of L and $n-i-1$ copies of K, will be denoted by $I_i(K, L)$. If $L = D$, then $I_i(K, D)$ is called the ith intersection body of K and is denoted by $I_i K$, $0 \leqslant i \leqslant n$ (see [15]). From the definition (2.8), it follows that the ith intersection body of K, $I_i K$ $(0 \leqslant i \leqslant n)$, is the centrally symmetric body (see [6]) whose radial function, for $u \in S^{n-1}$, is given by

$$\rho(I_i K, u) = \tilde{w}_i\bigl(K \cap u^\perp\bigr), \tag{2.9}$$

and $I_0 K = IK$.

Let $K_i \in \mathcal{S}^n$ $(1 \leqslant i \leqslant n-1)$. We define the dual mixed body $\{K_1, \ldots, K_{n-1}\}$ of $K_1, \ldots, K_{n-1}$ to be that star body whose radial function satisfies, for $u \in S^{n-1}$,

$$\rho\big(\{K_1, \ldots, K_{n-1}\}, u\big) = \big[\rho(K_1, u)\rho(K_2, u) \cdots \rho(K_{n-1}, u)\big]^{\frac{1}{n-1}}. \tag{2.10}$$

From the definition of radial function, it follows that $\{K_1, \ldots, K_{n-1}\}$ is symmetric in its arguments, and $\{\alpha K_1, \ldots, K_{n-1}\} = \alpha\{K_1, \ldots, K_{n-1}\}$. It also follows that $\{K, \ldots, K\} = K$.

We use the notation $\{L_1, i_1; \ldots; L_m, i_m\}$ for the dual mixed body in which L_j appears i_j times. In particular, we let $\{K, L\}_i$ denote the dual mixed body $\{K, \ldots, K, L, \ldots, L\}$, with i copies of L and $n-i-1$ copies of K. For the case where $L = D$, we write $\{K\}_i$ rather than $\{K, D\}_i$. We note that $\{K\}_0 = K$, while $\{K\}_{n-1} = D$.

From definition (2.10), together with (2.8), it follows that

$$I_0\{K\}_i = I_i K. \tag{2.11}$$

3. Proof of main results

The following results will be required to prove our main theorems.

Lemma 3.1 *(Busemann Intersection Inequality). (See [3, p. 333].) If $K \in \mathcal{S}^n$, then*

$$V(IK) \leqslant \frac{\omega_{n-1}^n}{\omega_n^{n-2}} V(K)^{n-1}, \tag{3.1}$$

with equality if and only if K is a centered ellipsoid.

In fact, taking $I_0 K = IK$ and $W_0(K) = V(K)$, inequality (B) is just the well-known Busemann intersection inequality (3.1) (see [3, p. 333]).

Lemma 3.2. *(See [11, p. 535].) If $K \in \mathcal{S}^n$ and $0 \leqslant i < j < n$, then*

$$\omega_n^i \tilde{W}_j(K)^{n-i} \leqslant \omega_n^j \tilde{W}_i(K)^{n-j}, \tag{3.2}$$

with equality if and only if K is a centered ball.

Lemma 3.3. *Let $K \in \mathcal{S}^n$ and $0 \leqslant i < n-1$. Then*

$$\tilde{W}_0\big(\{K\}_i\big)^{(n-1)(n-i)} \leqslant \omega_n^i \tilde{W}_i(K)^{n(n-1-i)}, \tag{3.3}$$

with equality if and only if K is a ball.

Proof. Since $V(\{K\}_i) = \tilde{V}(\{K\}_i, n-1; \{K\}_i)$, from (2.10) it follows that

$$V\big(\{K\}_i\big) = \tilde{V}\big(K, n-i-1; D, i; \{K\}_i\big).$$

The dual Aleksandrov–Fenchel inequality (2.3), with $L = \{K\}_i$, $s = n-i-1$ and $t = 1$, can be used to conclude that

$$V\big(\{K\}_i\big)^{n-i} \leqslant \tilde{V}(K, n-i; D, i)^{n-i-1} \tilde{V}\big(\{K\}_i, n-i; D, i\big).$$

However, from inequality (2.4) we have

$$\tilde{V}\big(\{K\}_i, n-i; D, i\big) \leqslant \omega_n^{\frac{i}{n}} V\big(\{K\}_i\big)^{\frac{n-i}{n}},$$

with equality if and only if $\{K\}_i$ is a ball, or equivalently, if and only if $\rho_{\{K\}_i} = \rho_K^{\frac{n-1-i}{n-1}} = \rho_{D'}$ for some ball D. It follows that K itself is a ball. If we combine the two inequalities above, we obtain the inequality (3.3). $\square$

Lemma 3.4. *If $K \in \mathcal{S}^n$ and $0 \leqslant i < n-1$, then*

$$\tilde{W}_{n-1}(I_i K) = \omega_{n-1} \tilde{W}_{i+1}(K). \tag{3.4}$$

Proof. For $K \in \mathcal{S}^n$, using definition (2.5), we have

$$\tilde{W}_{n-1}(K) = \frac{1}{n}\int_{S^{n-1}} \rho(K,u)\,dS(u).$$

Replace K with $I_i K$, and use definition (2.9), and observe that the formula in Lemma 3.4 is just the dual Kubota formula (see [3]):

$$\omega_{n-1}\tilde{W}_{i+1}(K) = \frac{1}{n}\int_{S^{n-1}} \tilde{w}_i\bigl(K \cap u^{\perp}\bigr)\,dS(u). \qquad \square$$

Proof of Theorem 1. From inequality (B), it follows that

$$\left(\frac{\omega_n^{n-1}}{\omega_{n-1}^n}\right)\tilde{W}_0(I_0K) \leqslant \omega_n \tilde{W}_0(K)^{n-1}, \tag{3.5}$$

with equality if and only if K is a centered ellipsoid.

In Lemma 3.2, let $i = 0$ and $j = n-1$ in inequality (3.2), to give

$$\tilde{W}_{n-1}(K)^n \leqslant \omega_n^{n-1}\tilde{W}_0(K),$$

with equality if and only if K is a centered ball.

In this inequality take I_0K for K, it follows that

$$\tilde{W}_{n-1}(I_0K)^n \leqslant \omega_n^{n-1}\tilde{W}_0(I_0K), \tag{3.6}$$

with equality if and only if K is a centered ball.

According to Lemma 3.4, we have

$$\tilde{W}_{n-1}(I_0K) = \omega_{n-1}\tilde{W}_1(K). \tag{3.7}$$

Combining inequality (3.6) and equality (3.7), we obtain

$$\tilde{W}_1(K)^n \leqslant \left(\frac{\omega_n^{n-1}}{\omega_{n-1}^n}\right)\tilde{W}_0(I_0K),$$

with equality if and only if K is a centered ball. The proof is complete. $\square$

Proof of Theorem 2. For $K \in \mathcal{S}^n$ and $0 \leqslant i < n-1$. Putting $j = n-1$ in inequality (3.2), it follows that,

$$\tilde{W}_{n-1}(K)^{n-i} \leqslant \omega_n^{n-1-i}\tilde{W}_i(K).$$

In this inequality take I_iK for K, use Lemma 3.4, to give

$$\tilde{W}_{i+1}(K)^{n-i} \leqslant \left(\frac{\omega_n^{n-1-i}}{\omega_{n-1}^{n-i}}\right)\tilde{W}_i(I_iK).$$

On the other hand, let $i = 0$ and $j = i$ in inequality (3.2), to yield

$$\tilde{W}_i(K)^n \leqslant \omega_n^i \tilde{W}_0(K)^{n-i},$$

from which it follows that

$$\tilde{W}_i(I_iK)^n \leqslant \omega_n^i \tilde{W}_0(I_iK)^{n-i}. \tag{3.8}$$

Take $\{K\}_i$ for K in (B), use formula (2.11), and the result is

$$\omega_n^{n-2}\tilde{W}_0(I_iK) \leqslant \omega_{n-1}^n \tilde{W}_0\bigl(\{K\}_i\bigr)^{n-1}. \tag{3.9}$$

When inequalities (3.8), (3.3) are combined with inequalities (3.9), the result is

$$\left(\frac{\omega_n^{n-1-i}}{\omega_{n-1}^{n-i}}\right)\tilde{W}_i(I_iK) \leqslant \omega_n \tilde{W}_i(K)^{n-i-1}.$$

The proof is complete. $\square$

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